A STAR-PRODUCT APPROACH TO NONCOMPACT QUANTUM GROUPS.

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Abstract:

Using duality and topological theory of well behaved Hopf algebras (as defined in [2]) we construct star-product models of non compact quantum groups from Drinfeld and Reshetikhin standard deformations of enveloping Hopf algebras of simple Lie algebras. Our star-products act not only on coefficient functions of finite-dimensional representations, but actually on all C^∞ functions, and they exist even for non linear (semi-simple) Lie groups.

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Introduction.

In the star-product approach to quantum mechanics [1], the observables are usual functions on phase space, so the geometry is not changed, but the product of observables is no longer the commutative one; the new product (star-product) is a deformation [6] of the initial one, with parameter $\hbar$ and leading cocycle the Poisson bracket. The simplest example is the Moyal product, which is completely explicit. By analogy, in his fundamental Berkeley paper on quantum groups [3], Drinfeld insisted on the similitude which should exist between star-product formulation of quantum mechanics, and "functional realizations" (cf. e.g.: [5] or [15]) of quantum groups: these realizations should be deformations of the commutative product of functions on the corresponding group, with (skew symmetrized) leading cocycle a Poisson bracket. In [2], this nice interpretation was given a precise mathematical justification (using adapted duality arguments and triviality results of Drinfeld [4]) in the case of compact Lie groups : it was shown that all standard "functional" models (e.g.: [5]) can be realized in such a way. Let us go further in the discussion: in quantum mechanics, the star-product is fortunately not limited to act on polynomial functions, but actually lives on C^∞-functions. On the other hand, all known "functional" models of quantum groups are given by generators and relations, so when realized as preferred deformations [7], as in [2], they live on coefficient functions of finite dimensional (f.d.) representations. This restriction seems unnatural, and only justified by necessity : FRT-models were initially defined by generators and relations! Actually, and this is one of the main results of [2], the models do extend to all C^∞-functions, as they should, and therefore the similitude with the star-product approach to quantum mechanics is complete. This being noted, the coefficient functions are a sub-Hopf algebra for the star-product, which can be seen as a dense core on which the star-product can be more easily computed.

In the present paper we want to show how the results of [2] for compact Lie groups can be extended to connected semi-simple Lie groups. This will answer various attempts to define "non-compact" quantum groups, and also give a serious framework for a discussion of real forms (see e.g.: [13]). To be precise, if one wants to construct a star-product only on the space of f.d. representations coefficients, then it is a straightforward improvement of [2], because we show that the algebra of coefficients of f.d. representations of

a connected semi-simple Lie group can always be identified with the same algebra for a suitable compact Lie group (an avatar of the Weyl unitary trick; as a consequence this algebra is a finitely generated domain). By the way, we show that the core of coefficient functions is dense in C^∞-functions if and only if the group is linear, a result which stresses the difference between the linear and non linear cases.

As mentionned above, in our opinion, the coefficient space is much too restrictive for star-product theory, and the good space to deal with is the space of C^∞-functions. So we show that in every case (linear or not) the star-product extends to all C^∞-functions: starting with a Hopf deformation of the enveloping algebra of a semi-simple complex Lie algebra $\mathfrak{g}$ (such as Drinfeld models [3], or Reshetikhin models [11]), we construct a preferred deformation [7] of the algebra of C^∞-functions on any Lie group with Lie algebra a real form of $\mathfrak{g}$. This is completely new for semi-simple Lie groups which are not linear, a case in which it is obviously meaningless to try and define quantum models by standard FRT-techniques of generators and relations. We think that this result will lead to new models defined by generators and relations, but acting on coefficients of infinite dimensional representations (such as the discrete series when it exists), and we intend to develop this aspect in a subsequent paper.

1. Hopf algebras of functions and distributions associated to a connected Lie group

Let G be a connected Lie group, and $H(G) = C^\infty(G)$. When endowed with its usual Fréchet topology, $H(G)$ is a nuclear Montel space; it is well known that $H(G \times G) = H(G)\hat{\otimes}H(G)$. Now, the commutative product on $H(G)$ defines a topological algebra structure. Moreover, setting $\delta(f)(x,y) = f(xy), f \in H(G), x, y \in G$, we get a continuous coproduct. There is a unit (the constant 1 function), a counit (the Dirac distribution ∂_e), and an antipode defined by $S_0(f)(x) = f(x^{-1}), f \in H(G), x \in G$. Since all the mappings involved are continuous, $H(G)$ is a well behaved topological Hopf algebra ([2] (1.2)).

Now let $A(G) = H(G)^*$ be the space of compactly supported distributions on G, with strong dual topology; by ([2] (1.3)), the transposition defines a well behaved topological Hopf algebra structure on $A(G)$: product is the

convolution product, the unit is the Dirac distribution ∂_e the counit is the evaluation on 1. In order to check the coproduct, we introduce $\partial : G \to A(G)$ defined by $< \partial_x \mid f >= f(x), f \in H(G), x \in G$; it is easily seen that ∂ actually defines a topological inclusion of G as a closed subset of A, so in the sequel we identify G and $\partial(G)$. This being done, one has $G^\perp = \{0\}$, so $\overline{\mathrm{Vect}(G)} = A(G)$, and the coproduct $\triangle_0 : A(G) \to A(G)\hat{\otimes}A(G) = A(G \times G)$ is given on G by $\triangle_0(x) = x \otimes x, x \in G$.

Let $\mathfrak{g}_0$ be the Lie algebra of G, $\mathfrak{g}$ its complexification, and $\mathcal{U}(\mathfrak{g})$ the corresponding enveloping algebra. We identify, as usual, $\mathcal{U}(\mathfrak{g})$ and the algebra of left invariant differential operators of finite order on G; the left regular representation of G on $H(G)$ defined by $L_g(f)(x) = f(g^{-1}x),\ f \in H(G), g, x \in G,$ is a C^∞ representation of G [14] and the same holds for its contragredient [14] :

$$< \check{L}_g(a) \mid f >=< a \mid L_{g^{-1}}(f) >, a \in A, f \in H, g \in G$$

. But $\check{L}_g(a) = \partial_g.a,$ so taking $a = \partial_e$ we see that $\partial : G \to A$ is a C^∞ mapping; we define a linear map $i : \mathcal{U}(\mathfrak{g}) \to A$ by $i(u)(f) = \partial_e[u(f)], u \in \mathcal{U}(g), f \in H(G)$. By left invariance one has $u(f)_g = i(u)[L_{g^{-1}}(f)]$, so i is one to one, and it is easy to check that i is a morphism, so we identify $\mathcal{U}$ and $i(\mathcal{U})$ and consider in the sequel that $\mathcal{U}(g) \subset A$. Since

$$< \frac{d}{dt}(\mathrm{exp}tX)_{t=0} \mid f >=< \frac{d}{dt}(\check{L}_{\mathrm{exp}t\ X}(\partial_e))_{t=0} \mid f >$$

$$= \frac{d}{dt} < \partial_e \mid L_{exp\ (-tX)}(f) >_{t=0} = X(f)_e = i(X)(f)$$

one has $\frac{d}{dt}(\mathrm{exp}t\ X)_{t=0} = X$, in $A(G)$. Therefore, by differentiation of

$$\triangle_0(\mathrm{exp}t\ X) = \mathrm{exp}tX \otimes \mathrm{exp}t\ X,$$

we deduce that $\triangle_0(X) = X \otimes 1 + 1 \otimes X, X \in \mathfrak{g}_0$. By the same technique, the antipode of $A(G)$ restricts on $\mathcal{U}(\mathfrak{g})$ to the usual antipode of $\mathcal{U}(\mathfrak{g})$, and finally $\mathcal{U}(\mathfrak{g})$ is a Hopf subalgebra of $A(G)$. Another interpretation of this result is as follows: $\mathcal{U}(\mathfrak{g})$ has a natural Hopf structure, which extends to a Hopf structure on $A(G)$, though $\mathcal{U}(\mathfrak{g})$ is certainly not dense in $A(G)$. This remark is the key to understand the star-product realization of quantum groups that we shall give later.

2. Hopf algebras of coefficients and generalized distributions associated to a connected semi-simple Lie group

Let G be a connected semi-simple Lie group. Given a finite dimensional (f.d.) representation π of G, and any $M \in \mathcal{L}(V_\pi)$, the coefficient of π associated to M is defined by $C^\pi_M(x) = \mathrm{Tr}(M\pi(x)), x \in G$, and is an analytic function on G. When π is irreducible, by Burnside theorem, $C^\pi : \mathcal{L}(V_\pi) \to H(G)$ is one to one, and we define $H(G)_\pi = C^\pi(\mathcal{L}(V_\pi))$. Then we fix once for all a set Π of f.d. irreducible representations of G which contains one and only one element of each equivalence class, and such that if $\pi \in \Pi$ and $\pi \neq \breve{\pi}$, then $\breve{\pi} \in \Pi$.

Let $\mathcal{H}(G)$ be the subspace of G generated by the coefficients of all f.d. representations.

(2.1) Lemma: $\mathcal{H}(G) = \bigoplus\limits_{\pi\in\Pi} H(G)_\pi$

Proof: Since $\mathfrak{g}$ is semi-simple, any representation of G is semi-simple, so $\mathcal{H}(G) = \Sigma_{\pi\in\Pi} H(G)_\pi$. Let $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}_0$, $\mathfrak{u}$ the associated compact real form of $\mathfrak{g}$, and U the compact simply connected group with Lie algebra $\mathfrak{u}$. Given $\pi \in \Pi$, $d\pi$ is a representation of $\mathfrak{g}$, therefore of $\mathfrak{u}$, so π is actually a representation of U, and C^π_M is also an analytic function on G. By the Peter-Weyl theorem $\Sigma_{\pi\in\Pi} H(U)_\pi$ is a direct sum in $H(U)$; assume now that $\Sigma_\pi f_\pi = 0, f_\pi = C^\pi_{M_\pi} \in H(G)_\pi$. Using Taylor formula, we deduce that $\Sigma_\pi f_\pi(X^n) = 0, \forall X \in \mathfrak{g}_0, \forall n \geq 0$, so $\Sigma_\pi f_\pi = 0$ on $\mathcal{U}(\mathfrak{g})$. Using once more Taylor formula and the analyticity of $\Sigma_\pi f_\pi$ on U, $\Sigma_\pi f_\pi = 0$ on U, so $f_\pi = 0$ on $U, \forall \pi$, therefore $M_\pi = 0, \forall\pi$, and then $f_\pi = 0$ on $G, \forall\pi$. ∎

Now $\mathcal{H}(G)$ is a Hopf subalgebra of $H(G)$, that we shall call the *Hopf algebra of coefficients of* G. This is seen as follows: As in ([2] (2.3)), one has: $C^\pi_M \otimes C^{\pi'}_{M'} = C^{\pi\otimes\pi'}_{M\otimes M'}$, if $\pi \otimes \pi'$ denotes the usual tensor product of the representations π and π' of G, so $\mathcal{H}(G)$ is a subalgebra. Given a basis $\{e_i\}$ of V_π, let $M = e^*_i \otimes e_j$, then $C^\pi_M(x) = \pi_{ij}(x), \Delta(\pi_{ij}) = \Sigma_k \pi_{ik} \otimes \pi_{kj}$, so $\delta(\mathcal{H}(G)) \subset \mathcal{H}(G) \otimes \mathcal{H}(G)$. It is clear that the unit of $H(G)$ is in $\mathcal{H}(G)$, and moreover the counit of $H(G)$ defines a counit on $\mathcal{H}(G)$. Finally, $S_0(\pi_{ij}) = \breve{\pi}_{ji}$, if $\breve{\pi}$ denotes the contragredient of π, so $S_0(\mathcal{H}(G)) \subset \mathcal{H}(G)$.

Finally, we note that since the elements of $\mathcal{H}(G)$ are analytic functions on G, $\mathcal{H}(G)$ *is actually a domain*. Moreover, with the notations of the proof of

Lemma (2.1), $\mathcal{H}(G) \subset \mathcal{H}(U)$, and it is well known that $\mathcal{H}(U)$ is of countable dimension, so the same holds for $\mathcal{H}(G)$. This last remark and ([2] (1.5.1)) show that, when endowed with its natural topology, $\mathcal{H}(G)$ is a well behaved topological Hopf algebra, so its dual $\mathcal{A}(G) = \prod_{\pi \in \Pi} \mathcal{L}(V_\pi)$ is also a topological Hopf algebra (with Fréchet product topology) for the transposed structure, if we define the duality (as in [2]) by $< \Sigma\, M_\pi \ \mid \ \Sigma\, C^\pi_{N_\pi} >= \Sigma\, \mathrm{Tr}(M_\pi\, N_\pi)$; then the product in $A(G)$ is given by:

$$\Sigma\, M_\pi \circ \Sigma M'_\pi = \Sigma\, M_\pi \ \circ \ M'_\pi, \ \ M_\pi, M'_\pi \in \mathcal{L}(V_\pi).$$

Clearly, any f.d. representation of G extends to a representation of $\mathcal{A}(G)$. To go further, we have to give some details about the inclusion $\mathcal{H}(G) \subset H(G)$, which is actually a continuous map. This is done in the following proposition:

(2.2) Proposition: *$\mathcal{H}(G)$ is dense in $H(G)$ if and only if G has a faithful finite dimensional representation.*

Remark : By a classical result of Harish Chandra ([8], or see the proof of (2.4)), the condition of (2.2) is equivalent to the fact that $\mathcal{H}(G)$ seperates between points of G. If this condition is satisfied, by a result of Goto [14], G being a semi-simple analytic subgroup of some $GL(p, \mathbb{C})$ is actually closed in that $GL(p, \mathbb{C})$, so we can consider that $G \subset GL(p, \mathbb{C})$, with the induced topology; in the sequel we shall call such a group a semi-simple linear group. Note that a semi-simple linear group has finite center, but the converse is false (e.g.: the metapletic covering of $SL(2, \mathbb{R})$).

Proof of (2.2): If $\mathcal{H}(G)$ is dense in $H(G)$, apply the Harish Chandra result. Conversely if G is a closed linear subgroup of $GL(p, \mathbb{C})$, by a standard partition of unity argument, any C^∞ function on G can be extended to a C^∞ function on $GL(p, \mathbb{C})$. But $GL(p, \mathbb{C})$ has a global chart, namely if $M = (m_{ij}) \in GL(p, \mathbb{C}), (x_{ij} = \mathrm{Re}(m_{ij}), y_{ij} = \mathrm{Im}(m_{ij}))$ defines a global coordinate system. By a slight and well known improvement of the Stone-Weierstrass theorem ([12]), on any open set of $\mathbb{R}^{p^2}$, polynomial functions are dense in C^∞ functions, so (2.2) is true for $GL(p, \mathbb{C})$ (though it is not semi-simple). By restriction to G, polynomials in coefficients of the natural representation of G and of its complex conjugate are dense in $C^\infty(G)$. ∎

So when G is semi-simple and linear, from the inclusion $\mathcal{H}(G) \subset H(G)$ and the density of $\mathcal{H}(G)$ in $H(G)$, we can deduce an inclusion of $A(G)$ in

$\mathcal{A}(G)$. Since G and $\mathcal{U}(\mathfrak{g})$ are contained in $A(G)$, it results, in that case, that $G \subset \mathcal{A}(G)$ and $\mathcal{U}(\mathfrak{g}) \subset \mathcal{A}(G)$. Moreover one has $\overline{\mathrm{Vect}(G)} = \mathcal{A}(G)$. Now it is natural to ask what happens when G is no longer linear, and the answer is given by:

(2.3) Proposition: *One has $\mathcal{U}(\mathfrak{g}) \subset \mathcal{A}(G)$, and $\overline{\mathcal{U}(\mathfrak{g})} = \mathcal{A}(G)$.*

Proof: Let G' be the adjoint group of G, which is a quotient of G and has the same Lie algebra. The set Π' of irreductible f.d. of G' is a subset of the set Π of irreductible f.d. representations of G. So if $u \in \mathcal{U}(\mathfrak{g})$ satisfies $\pi(u) = 0, \forall \pi \in \Pi$, since G' is semi-simple linear one has $u = 0$, which proves that $\mathcal{U}(\mathfrak{g}) \subset \mathcal{A}(G)$. Finally the algebra $\mathcal{A}(G)$ is the bicommutant of the semi-simple representation $\oplus_{\pi\in\Pi}\pi$ of $\mathcal{U}(\mathfrak{g})$, so density follows from the Jacobson density theorem. ∎

(2.4) Remarks: (1) When G is not linear one has a map $\oplus_{\pi\in\Pi}\pi : G \to \mathcal{A}(G)$, with dense range, but which is not one to one.

(2) When G is linear, the inclusion $\mathcal{H}(G) \subset H(G)$ with dense range leads, by transposition ([2] (1.3)), to an inclusion of topological Hopf algebras $A(G) \subset \mathcal{A}(G)$; it results for instance that the coproduct of $A(G)$ is the restriction of the coproduct of $\mathcal{A}(G)$, and so, as proved in (section 1), one has $\triangle_0(x) = x \otimes x, x \in G$, and $\triangle_0(X) = X \otimes 1 + 1 \otimes X, X \in \mathfrak{g}$. So, as in the case of $A(G)$, the Hopf structure of $\mathcal{A}(G)$ is the extension of the natural Hopf structure of $\mathcal{U}(\mathfrak{g})$. From the proof of (2.3), this last result holds also if G is not linear.

(3) Let us note that $H(G) = C^\infty(G)$ completely specifies G, in the following sense: G is exactly the set of group-like elements of $A(G) = H(G)^*$, and the topology of G is inherited from the topology of A. On the other hand, when G is not linear, G cannot be contained in $\mathcal{A}(G)$ (because of the abovementioned result by Harish Chandra). Actually $\mathcal{H}(G)$ and $\mathcal{A}(G)$ are related to a compact group, as shown by:

(2.5) Proposition: *There exists a compact connected Lie group K, with Lie algebra the compact real form of $\mathfrak{g}$ associated to $\mathfrak{g}_0$, such that $\mathcal{H}(G) = \mathcal{H}(K)$.*

Corollary: *$\mathcal{H}(G)$ is a finitely generated algebra.*

Proof: Since $\mathcal{H}(G\times G') = \mathcal{H}(G)\otimes\mathcal{H}(G')$, we can assure that G is simple. Let $\mathfrak{g}_0 = \mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}_0$, $\mathfrak{u} = \mathfrak{k}+i\mathfrak{p}$ the associated compact real form of $\mathfrak{g}$, and U the simply connected compact simple group with Lie algebra $\mathfrak{u}$. Any f.d. representation ρ of G is actually also a representation of U, since $d\rho$ extends to $\mathfrak{g}$, and equivalence and irreducibility are preserved. Assume ρ non trivial. Let $K_\rho = \rho(U)$, by the same argument, any f.d. representation θ of K_ρ is actually also a representation of $\tilde{G}$, the universal covering of G. Let us show that θ is in fact a representation of G: since ρ is a faithful representation of K_ρ, which is compact, $\mathcal{H}(K_\rho)$ is generated by the coefficients of ρ and $\breve{\rho}$ [2] ; equivalently, θ is a subrepresentation of a sum of tensor products of ρ and $\breve{\rho}$. Let us denote this situation by $\theta\subset \mathrm{Pol}_\otimes(\rho,\breve{\rho})$. But $\gamma =$ $\mathrm{Pol}_\otimes(\rho,\breve{\rho})$ is a representation of G, and $\theta\subset\gamma$ means that on some subspace W, one has $\gamma \mid K \mid W = \theta \mid K$; therefore $d\gamma(X)\mid_W = d\theta(X), X\in\mathfrak{g}$, and the integration of $d\theta$ leads to a representation of G. So any representation of K_ρ is actually a representation of G. Now $K_\rho = U/\mathcal{Z}_\rho$, where $\mathcal{Z}_\rho$ is a subgroup of the center of U, which is finite, so when ρ varies over all f.d. non trivial representations of G, only a finite number of K_ρ will appear. We denote them by $K_1, ..., K_n$, and by $\rho_1, ..., \rho_n$ the associated representations of G.

Given a non trivial representation ρ of G, there exists K_i such that ρ is a faithful representation of K_i, and ρ, as a representation of K_i, is contained in some $\mathrm{Pol}_\otimes(\rho_i,\breve{\rho}_i)$ [2] which is a representation of G. Therefore the coefficients of ρ, as functions on G, are polynomials in the coefficients of ρ_i and $\breve{\rho}_i$. It results that $\mathcal{H}(G)$ is generated, as an algebra, by the coefficients of $\xi = \rho_1\oplus....\oplus\rho_n\oplus\breve{\rho}_1\oplus....\oplus\breve{\rho}_n$. Also if $N(G) = \bigcap_{\rho \text{ f.d.}} \rho$, one has $N(G) = \ker\xi$, which is the result of Harish Chandra quoted in (2.2). Now ξ is also a representation of $K = U/\bigcap_i \mathrm{Ker}\rho_i$, and each ρ_i is a representation of K. Since ξ is a faithful self dual representation of K which is compact, any representation of K is contained in some $\mathrm{Pol}_\otimes(\xi)$, so any representation of K is actually a representation of G. On the other hand the representations ρ_i and $\breve{\rho}_i$, which generate by tensor products all representations of G, are representations of K. This provides a one to one mapping from $\mathrm{Rep}(G)$ onto $\mathrm{Rep}(K)$, preserving equivalence and irreducibility, so $\Pi_G = \Pi_K = \Pi$, and then an isomorphism of Hopf algebras $\mathcal{H}(G)\simeq\mathcal{H}(K)$ is defined by: If $\pi\in\Pi, M\in\mathcal{L}(V_\pi)$, define $\phi_\pi(C^\pi_M \text{as a function on } G) = (C^\pi_M \text{ as a function on } K)$, and then $\phi = \Sigma_\Pi\phi_\pi$. ∎

(2.6) Remarks: (1) In the case of a compact group K, any f.d. faithful and self dual representation π is a complete set of representations of K (i.e. the coefficients of π generate $\mathcal{H}(K)$), and such a representation always exist (see [2] and references quoted therein). For a noncompact semi-simple linear group G, from the proof of (2.5), there exists a f.d. faithful and self dual representation π which is a complete set, but is has to be noted that any f.d. faithful and self dual representation needs not to be a complete set (see e.g. [9] p.116). Nevertheless, by the proof of (2.2), the coefficients of any f.d. faithful representation and of its conjugate always generate a dense subalgebra of $C^\infty(G)$. So one has to be careful !

(2) (2.5) is a reformulation of the Weyl unitary trick. Equivalent formulations using complex algebraic groups instead of compact Lie groups can be stated.

3. Star-products on $\mathcal{H}(G)$ and $H(G)$.

We keep the notations and assumptions of Section 2. In particular, G is always assumed to be a semi-simple connected Lie group. Let us consider a deformation $\mathcal{U}_t$ of $\mathcal{U} = \mathcal{U}(\mathfrak{g})$. As shown by Drinfeld [4], it is automatically trivial, so we can assume $\mathcal{U}_t = \mathcal{U}[[t]]$ with trivial product. Assume now that we have a coassociative coalgebra deformation of $\mathcal{U}_t$, with coproduct $\tilde{\Delta}$. By the argument of ([2] (6.2.1)), the initial counit ε is still a counit for the new structure, and by [7], there is an antipode, so we get finally a Hopf deformation. By [4], the new coproduct $\tilde{\Delta}$ is obtained from the initial one Δ_0 by a twist, i.e. there exists $\tilde{P} \in \mathcal{U} \otimes \mathcal{U}[[t]]$ such that $\tilde{\Delta} = \tilde{P}\Delta_o\tilde{P}^{-1}$. Set $\mathcal{A} = \mathcal{A}(G)$, and $A = A(G)$.

(3.1) Lemma: *The formula $\tilde{\Delta} = \tilde{P}\Delta_0\tilde{P}^{-1}$ defines a coassociative coproduct on $\mathcal{A}[[t]]$ and$A[[t]]$.*

Proof: Since $\mathcal{U}$ is contained in A and $\mathcal{A}$, $\tilde{\Delta}$ is clearly a morphism, and we have to show that it is still coassociative.

Let us start with a linear G. Then the proof is as in ([2](6.2.1)): one has $\mathcal{U} \subset A \subset \mathcal{A}$ and $\overline{\mathcal{U}} = \mathcal{A}$, and since the continuous maps $(\tilde{\Delta} \otimes I) \circ \tilde{\Delta}$ and $(I \otimes \tilde{\Delta}) \circ \tilde{\Delta}$ coincide on $\mathcal{U}$, they coincide on $\mathcal{A}$, and a fortiori on A.

Then we treat the case of general G. We introduce K as in (2.5): $\mathcal{A}(G) = \mathcal{A}(K)$, and we are back to the linear case. To get the result for A, we use

the density of Vect(G) in A as follows. Using the twist, our result will be proved if we prove the following: if formula $(\Delta_0 \otimes I) \circ \Delta_0 = \tilde{\phi} \circ (I \otimes \Delta_0) \circ \Delta_0 \tilde{\phi}^{-1}$, for some $\tilde{\phi} \in \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}[[t]]$, holds on $\mathcal{U}$, then it holds on A. To prove that, we set $\pi_x = (\Delta_0 \otimes I) \circ \ \Delta_0 \ (x), \pi'_x = \tilde{\phi} \circ (I \otimes \Delta_0) \circ \Delta_0(x) \circ \tilde{\phi}^{-1}$, and obtain two continuous morphisms from A into $A \hat{\otimes} A \hat{\otimes} A[[t]]$.

Using the formula $\frac{d}{d\tau}(\exp\tau X) = X \cdot \exp\tau X, X \in \mathfrak{g}$, we have:

$$\frac{d}{d\tau}(\pi(\exp\tau X)) = \pi(X)\pi(\exp\tau X) \text{ and } \frac{d}{d\tau}(\pi'(\exp\tau X)) = \pi(X)\pi'(\ \exp\tau X).$$

Thus $\frac{d}{d\tau}(\pi((\exp\tau X))^{-1}\pi'(\exp\tau X)) = 0$, so $\pi'(\exp\tau X) = \pi(\exp\tau X)$. But G is connected, π and π' are morphisms, so we obtain that $\pi'(x) = \pi(x), \forall x \in G$. From $\overline{\text{Vect}(G)} = A$, we conclude that $\pi = \pi'$. ∎

In order to have a Hopf structure on $\mathcal{A}[[t]]$, or $A[[t]]$, with coproduct $\tilde{\Delta}$, we need a counit and an antipode. For the counit, the initial one still works, and the proof is the same as ([2] (4.2.6)). For the antipode, by [7], it does exist, and the next problem is to show that it is an extension of the new antipode of $\mathcal{U}_t$. In the case of $\mathcal{A}[[t]]$, by (2.4) we can assume that the group is compact, and then use ([2] (6.2.1)), so the result is true.

In order to prove the same for $A[[t]]$, we have to clarify the relations between representations of G, $A(G)$ and $\mathcal{A}(G)$, as follows: any given f.d. representation π of G, up to equivalence, splits into a direct sum of representations in Π, which are all by definition representations of $\mathcal{A}(G)$, so it extends to $\mathcal{A}(G)$. Moreover, $\pi \in C^\infty(G, \mathcal{L}(V_\pi)) = C^\infty(G)\hat{\otimes}\mathcal{L}(V_\pi) = \mathcal{L}(A(G), \mathcal{L}(V_\pi))$, so π extends to a continuous linear map (still denoted by π) from $A(G)$ into $\mathcal{L}(V_\pi)$. It is well known that π is a morphism of algebras (cf. e.g. [14]), so it defines a representation of $A(G)$ in V_π. Obviously, equivalence and irreducibility are preserved. On the other hand the inclusion $\mathcal{H}(G) \subset H(G)$ provides, by transposition, a continous linear morphism from $A(G)$ into $\mathcal{A}(G)$, so any representation of $\mathcal{A}(G)$ is a representation of $A(G)$; finally, any representation of $A(G)$ defines, by restriction, a representation of G. So we have proved that the f.d. representations of G, $A(G)$, or $\mathcal{A}(G)$ are the same.

(3.2) Lemma: *The new antipode of $A[[t]]$ is an extension of the new antipode of $\mathcal{U}[[t]]$.*

Proof: Let us denote by $S_{\mathcal{U}}, S_A$ and $S_{\mathcal{A}}$ the respective new antipodes of $\mathcal{U}[[t]], A[[t]]$ and $\mathcal{A}[[t]]$. Given a f.d. irreductible representation π, we consider

$\tilde{\pi} = {}^{T}\pi.S_A$ and by restriction to $\mathfrak{g}$ we get a deformation of the representation $\breve{\pi}$ of $\mathfrak{g}$, which is trivial, since $\mathfrak{g}$ is semi-simple [10]. From this remark and Burnside theorem we deduce that there exists $u_\pi \in \mathcal{U}[[t]]$ such that :

$$\tilde{\pi}(u) = \breve{\pi}(u_\pi \cdot u \cdot u_\pi^{-1}), \ \forall u \in \mathcal{U}.$$

We set $a = \Sigma_{\pi\in\Pi}\, u_\pi \in \mathcal{A}[[t]]$ and use the last formula to get :

$$\pi(S_A(u)) = \pi(S_0(a)^{-1}S_0(u)S_0(a)), \forall u \in \mathcal{U}, \forall \pi \in \Pi.$$

Let $S' = S_0(a)^{-1} \cdot S_0 \cdot S_0(a)$, denote by m the product of $A[[t]]$ or $\mathcal{A}[[t]]$, and write $\tilde{\Delta}(u) = \Sigma_i \alpha_i \otimes \beta_i, u \in \mathcal{U}$, then :

$$\pi[m\circ(S'\otimes Id)\circ\tilde{\Delta}(u)] = \Sigma_i \pi(S_A(\alpha_i))\pi(\beta_i) = \pi[m\circ(S_A\otimes Id)\circ\tilde{\Delta}(u)] = \pi[\varepsilon(u)1],$$

$\forall u \in \mathcal{U}, \pi \in \Pi$. Since Π separates points on $\mathcal{U}$ by (2.3), we get on $\mathcal{U}$, and by continuity on $\mathcal{A}$, since $\overline{\mathcal{U}} = \mathcal{A}$:

$$m \circ (S' \otimes Id) \circ \tilde{\Delta} = \varepsilon.1$$

Therefore S' is an antipode for $\mathcal{A}[[t]]$ with its new coproduct, so using unicity of the antipode, $S' = S_{\mathcal{A}}$ but since $S_{\mathcal{A}}$ extends $S_{\mathcal{U}}$, we have $S' \mid \mathcal{U} = S_{\mathcal{U}}$, and therefore:

$$\pi(S_A(u)) = \pi(S_{\mathcal{U}}(u)), \forall u \in \mathcal{U}, \forall \pi \in \Pi.$$

Finally $S_A(u) - S_{\mathcal{A}}(u) \in \bigcap_{\pi\in\Pi} \mathrm{Ker}\pi \bigcap \mathcal{U} = \{0\}$. ■

We can now state our main result :

(3.3) Theorem: *There exists a topological Hopf deformation of $A(G)$ (resp: $\mathcal{A}(G)$) which extends the Hopf deformation $\mathcal{U}_t$ on $\mathcal{U}$.*

Now we apply the duality argument ([2] (3.8)):

Corollary: *The Hopf deformation $\mathcal{U}_t$ produces a preferred (i.e. unchanged coproduct and counit) topological Hopf deformation of $\mathcal{H}(G)$ and $H(G)$.*

The new product on $H(G)$ and $\mathcal{H}(G)$ is the desired star product. The above corollary applies for instance to the Drinfeld standard models [3], and also to the Reshetikhin models [11]. It can be seen that the deformation of

$\mathcal{H}(G)$ is the restriction of the deformation of $H(G)$, as in the compact case [2]. In the linear case, if we start with the Drinfeld standard model, then there exists a universal R-matrix satisfying Q.Y.B. equation [3], and with the very proof given in [2], the star product on $\mathcal{H}(G)$ satisfies relations of the type $RT_1T_2 = T_2T_1R$.

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